Limit theorems for multipower variation in the presence of jumps

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Abstract

In this paper we provide a systematic study of how the the probability limit and central limit theorem for realised multipower variation changes when we add finite activity and infinite activity jump processes to an underlying Brownian semimartingale.

Keywords: Bipower variation; Infinite activity; Multipower variation; Power variation; Quadratic variation; Semimartingales; Stochastic volatility.

1 Introduction

Multipower variation is the probability limit of normalised partial sums of powers of lags of absolute high frequency increments of a semimartingale as the sampling frequency goes to infinity. It was introduced by Barndorff-Nielsen and Shephard (2003,2004b,2004a,2006) in a series of papers motivated by some problems in financial econometrics. Realised multipower variation estimates this limit process and was shown, by Barndorff-Nielsen, Graversen, Jacod, Podolskij, and Shephard (2005), to reveal integrated volatility powers in general Brownian semimartingales. These authors also derived the corresponding central limit theory. Some detailed discussion of the econometric uses of these results are given in Barndorff-Nielsen, Graversen, Jacod, and Shephard (2006). Such continuous sample path limit processes are of interest in themselves, however Barndorff-Nielsen and Shephard were also interested in realised multipower variation as they showed it has some features which are robust to finite activity jump processes (i.e. jump components with finite numbers of jumps in finite time). In this paper we return to that issue, sharpening the results in the finite activity case and giving an analysis of the case where there are an infinite number of jumps. For a closely related analysis see Woerner (2006).

Specifically, we ask two new questions: (i) do these kinds of robustness results also hold when the jump process has infinite activity, (ii) is it possible to construct central limit theorems for realised multipower variation processes when there are jumps? In Section 2 of the paper we establish notation and provide various definitions. This is followed in Section 3 with an analysis of multipower variation in the case where the processes are Brownian semimartingales plus jumps. In Section 4 we specialise the discussion to the case where the jumps are Lévy or OU processes.

2 Multipower Variation (MPV)

Let X be an arbitrary stochastic process. Then the realised multipower variation (MPV) of X is based on increments, recorded every $\delta > 0$ time periods,

$$x_j = X_{j\delta} - X_{(j-1)\delta}, \quad j = 1, 2, ..., \lfloor t/\delta \rfloor .$$

It can be defined via the unnormalised version

$$[X_\delta]_t^{[\mathbf{r}]} = [X_\delta]_t^{[r_1,...,r_m]} = [X_\delta, ..., X_\delta]^{[r_1,...,r_m]} = \sum_{j=m}^{\lfloor t/\delta \rfloor} |x_{j-m+1}|^{r_1} \cdots |x_j|^{r_m},$$

or through its normalised version

$$\{X_\delta\}_t^{[\mathbf{r}]} = \{X_\delta\}_t^{[r_1,...,r_m]} = \delta^{1-r_+/2}[X_\delta]_t^{[\mathbf{r}]},$$

where $\mathbf{r}$ is short for $r_1, ..., r_m$ and $r_+ = \sum_{j=1}^m r_j$. It will be convenient to write $\max r = \max\{r_1, , , , r_m\}$. Similarly, for arbitrary processes $X^{(1)}, ..., X^{(m)}$ we let

$$[X_\delta^{(1)}, ..., X_\delta^{(m)}]_t^{[\mathbf{r}]} = \sum_{j=m}^{\lfloor t/\delta \rfloor} |x_{j-m+1}^{(1)}|^{r_1} \cdots |x_j^{(m)}|^{r_m},$$

while we always assume that $r_j \geq 0$ and $r_+ > 0$.

3 MPVCiP and MPVCLT for $\mathcal{BSM}$ + jump process

Brownian semimartingales (denoted $\mathcal{BSM}$) are defined as the class of continuous semimartingales

$$Y_t = \int_0^t a_u \mathrm{d}u + \int_0^t \sigma_u \mathrm{d}W_u, \tag{1}$$

where a is predictable, W is standard Brownian motion and σ is càdlàg.

We say that the Brownian semimartingale Y satisfies CiP (converges in probability) for MPV (denoted MPVCiP) provided that

$$\{Y_\delta\}_t^{[\mathbf{r}]} \xrightarrow{p} d_{\mathbf{r}} \sigma_t^{r_+*} = d_{\mathbf{r}} \int_0^t \sigma_u^{r_+} \mathrm{d}u,$$

where $d_{\mathbf{r}}$ is a known constant depending only on $\mathbf{r}$.

We say that Y satisfies the central limit theorem (CLT) for MPV (denoted MPVCLT) provided

$$\delta^{-1/2}\left(\{Y_\delta\}_t^{[\mathbf{r}]} - d_{\mathbf{r}}\sigma_t^{r_+*}\right) \stackrel{law}{\to} c_{\mathbf{r}} \int_0^t \sigma_u^{r_+} \mathrm{d}B_u$$

where B is a Brownian motion, $Y \perp\!\!\!\perp B$ (i.e. Y is independent of B), and $c_{\mathbf{r}}$ is a known constant depending only on $\mathbf{r}$. Under some mild additional assumptions on the σ process such a CLT holds, see Barndorff-Nielsen, Graversen, Jacod, Podolskij, and Shephard (2005).

We will now study what happens to the limiting distribution when we add jumps to Y. The only existing results we know of are due to Jacod and Protter (1998) who studied the case where $\mathbf{r} = 2$, $Y \in \mathcal{BSM}$ and the jumps come from a purely discontinuous Lévy process, and Woerner (2006) who derives closely related results to ours. Thus we shall discuss extensions of MPVCiP and MPVCLT for $\mathcal{BSM}$ to processes of the form

$$X = Y + Z$$

where $Y \in \mathcal{BSM}$ while Z is a process exhibiting jumps.

We assume that Y satisfies MPVCiP or MPVCLT and consider to which extent this behaviour remains the same when Z is added to Y, i.e. whether the influence of Z is negligible (in this respect). When it is negligible we say that MPVCiP or MPVCLT holds for X. Thus we ask whether:

(i) For the CiP case

$$\delta^{1-r_+/2}\left([X_\delta, ..., , X_\delta]^{[r_1,...,r_m]} - [Y_\delta, ..., , Y_\delta]^{[r_1,...,r_m]}\right) = o_p(1).$$

(ii) For the CLT case

$$\delta^{1-r_+/2}\left([X_\delta, ..., , X_\delta]^{[r_1,...,r_m]} - [Y_\delta, ..., , Y_\delta]^{[r_1,...,r_m]}\right) = o_p(\delta^{1/2}).$$

We shall use the following fact

Lemma 1 *The Brownian semimartingale Y satisfies, uniformly in j,*

$$\delta^{-1/2}|Y_{j\delta} - Y_{(j-1)\delta}| = O_p(|\log\delta|^{1/2}). \tag{2}$$

Proof. First we split

$$|Y_{j\delta} - Y_{(j-1)\delta}| \leq \left|\int_{(j-1)\delta}^{j\delta} a_u \mathrm{d}u\right| + \left|\int_{(j-1)\delta}^{j\delta} \sigma_u \mathrm{d}W_u\right|$$

and note that the first part is $O_p(\delta)$ whereas, by the Dubins-Schwarz theorem,

$$\int_0^t \sigma_u \mathrm{d}W_u = B_{\int_0^t \sigma_s^2 \mathrm{d}s}$$

for a standard Brownian motion B. Lévy's theorem on the uniform modulus of continuity of Brownian motion states that

$$\mathrm{P}\left(\limsup_{\varepsilon\downarrow 0}\left(\sup_{0\leq t_1<t_2\leq T: t_2-t_1\leq\varepsilon} \frac{|B_{t_2} - B_{t_1}|}{\sqrt{2\varepsilon|\log(\varepsilon)|}}\right) = 1\right) = 1.$$

Since

$$\int_{t_1}^{t_2} \sigma_s^2 \mathrm{d}s \leq |t_2 - t_1| \sup_{0 \leq s \leq T} \sigma_s^2$$

and the latter supremum is a.s. finite, we deduce that, as required,

$$\mathrm{P}\left(\limsup_{\varepsilon \downarrow 0} \left(\sup_{0 \leq t_1 < t_2 \leq T : t_2 - t_1 \leq \varepsilon} \frac{|Y_{t_2} - Y_{t_1}|}{\sqrt{2\varepsilon |\log(\varepsilon)|}} \right) < \infty \right) = 1.$$

■

Without the sup over t_1 and t_2, for fixed t, the result holds with log replaced by log log.

3.1 Finite activity case

We first perturb a suitable $Y \in \mathcal{BSM}$ for which MPVCiP (and/or MPVCLT) holds by a finite activity jump process Z, not necessarily independent of Y.

Proposition 1 *When Z is a finite activity jump process, (i) MPVCiP holds if* $\max r < 2$, *(ii) MPVCLT holds if* $\max r < 1$.

Proof. Consider the m-th order MPV process $[X_\delta]^{[\mathbf{r}]}$. Pathwise, the number of jumps of Z is finite and, for sufficiently small δ, none of the additive terms in $[X_\delta, ..., X_\delta]^{[r_1,\dots,r_m]}$ involves more than one jump. Each of the terms in $[X_\delta, ..., X_\delta]^{[r_1,\dots,r_m]}$ that contains no jumps are of order $O_p\left((\delta |\log \delta|)^{r_+/2}\right)$. Any of the terms that do include a jump is of order $O_p\left((\delta |\log \delta|)^{(r_+ - \max r)/2}\right)$. Hence

$$\begin{aligned} \delta^{1-r_+/2}([X_\delta]^{[r]} - [Y_\delta]^{[r]}) &= \delta^{1-r_+/2} O_p((\delta |\log \delta|)^{(r_+ - \max r)/2}) \\ &= O_p(\delta^{1-\max r/2} |\log \delta|^{(r_+ - \max r)/2}). \end{aligned}$$

So CiP is not influenced by Z so long as $\max r < 2$, while CLT continues to hold if $\max r < 1$. ■

The bounds $\max r < 2$ and $\max r < 1$ are tight conditions. If the equality was to hold, we get discontinuous distributional limits. If the inequalities are reversed, limits jump to infinity at the first jump time of Z, except in trivial cases.

The above CLT result is of some importance. It means that we can use multipower variation to make mixed Gaussian inference about $\int_0^t \sigma_u^2 \mathrm{d}u$, integrated variance, in the presence of finite activity jumps processes so long as $\max r < 1$ and $r_+ = 2$. An example of this is where $m = 3$ and we take $r_1 = r_2 = r_3 = 2/3$ (that is using Tripower Variation (TPV)).

3.2 Infinite activity (IA) case

We start by establishing an inequality for MPV. Let a, b, c etc. denote arbitrary real numbers with $a + b = c$. The classical inequality

$$\left| \sum_{j=1}^n |a_j|^r - \sum_{j=1}^n |b_j|^r \right| \leq \sum_{j=1}^n |c_j|^r, \tag{3}$$

which holds for $0 < r \leq 1$, implies that if $\max r \leq 1$ then

$$\begin{aligned} & \left|[X_\delta, ..., X_\delta]^{[r_1,...,r_m]} - [Y_\delta, ..., Y_\delta]^{[r_1,...,r_m]}\right| \\ \leq \; & [Z_\delta, ..., Z_\delta]^{[r_1,...,r_m]} + [Z_\delta, ..., Z_\delta, Y_\delta]^{[r_1,...,r_m]} \; [\tbinom{m}{1}] \\ + \; & [Z_\delta, ..., Z_\delta, Y_\delta, Y_\delta]^{[r_1,...,r_m]} \; [\tbinom{m}{2}] + \cdots \\ + \; & [Z_\delta, Y_\delta, Y_\delta, ..., Y_\delta]^{[r_1,...,r_m]} \; [\tbinom{m}{m-1}] \end{aligned} \tag{4}$$

where the binomial coefficients indicate the relevant number of similar terms.

In the following we shall mostly restrict consideration to the case $r_1 = \cdots = r_m = r$.

3.2.1 Convergence in probability

For MPVCiP it suffices that the following conditions are met:

$$\delta^{1-mr/2}[Z_\delta, ..., Z_\delta]^{[r,...,r]} = o_p(1), \tag{5}$$

$$\delta^{1-(m-1)r/2}|\log\delta|^{r/2}[Z_\delta, ..., Z_\delta]^{[r,...,r]} \; [\tbinom{m}{1}] = o_p(1), ... \tag{6}$$

$$\delta^{1-r/2}|\log\delta|^{(m-1)r/2}[Z_\delta]^{[r]} \; [\tbinom{m}{m-1}] = o_p(1). \tag{7}$$

To show this we need to distinguish between the cases $0 < r \leq 1$ and $r > 1$.

When $0 < r \leq 1$ we have, by (4),

$$\begin{aligned} & \delta^{1-mr/2}\left|[X_\delta, ..., X_\delta]^{[r,...,r]} - [Y_\delta, ..., Y_\delta]^{[r,...,r]}\right| \\ \leq \; & \delta^{1-mr/2}[Z_\delta, ..., Z_\delta]^{[r,...,r]} + \delta^{1-(m-1)r/2}[Z_\delta, ..., Z_\delta, \delta^{-1/2}Y_\delta]^{[r,...,r]} \; [\tbinom{m}{1}] \\ + \; & \delta^{1-(m-2)r/2}[Z_\delta, ..., Z_\delta, \delta^{-1/2}Y_\delta, \delta^{-1/2}Y_\delta]^{[r,...,r]} \; [\tbinom{m}{2}] + \cdots \\ + \; & \delta^{1-r/2}[Z_\delta, \delta^{-1/2}Y_\delta, ..., \delta^{-1/2}Y_\delta]^{[r,...,r]} \; [\tbinom{m}{m-1}]. \end{aligned} \tag{8}$$

and the sufficiency of (5)-(7) follows.

For $r > 1$ we have

$$\left|\left(\delta^{1-mr/2}[X_\delta, ..., X_\delta]^{[r,...,r]}\right)^{1/r} - \left(\delta^{1-mr/2}[Y_\delta, ..., Y_\delta]^{[r,...,r]}\right)^{1/r}\right| \leq \left(\delta^{1-mr/2}\mathbf{S}\right)^{1/r}$$

where, in a compact notation,

$$\mathbf{S} = \sum_{j=m}^{\lfloor t/\delta \rfloor}\left|\sum_{\omega}\prod y_k \prod z_l\right|^r$$

and

$$\sum_{\omega}\prod y_k \prod z_l = (y_{j-m+1} + z_{j-m+1})\cdots(y_j + z_j) - y_{j-m+1}\cdots y_j,$$

where ω runs over all selections of one factor from each of the parentheses in the above equation, except the one leading to $y_{j-m+1}\cdots y_j$.

Now, if

$$\delta^{1-mr/2}\mathbf{S} = o_p(1) \tag{9}$$

then, on account of the previously established fact that $\delta^{1-mr/2}[Y_\delta, ..., Y_\delta]^{[r,...,r]}$ converges in probability to a positive random variable, we can conclude from the Minkovsky inequality that

$$(\left(\delta^{1-mr/2}\right)^{1/r}\left(\left([X_\delta, ..., X_\delta]^{[r,...,r]}\right)^{1/r} - \left([Y_\delta, ..., Y_\delta]^{[r,...,r]}\right)^{1/r}\right) = o_p(1).$$

To determine a sufficient condition for (9), and hence for MPVCiP, we note that in view of the inequality $|b+c|^r \le 2^{r-1}(|b|^r + |c|^r)$ there exists a constant C such that

$$|\sum_{\omega} \prod y_k \prod z_l|^r \le C \sum_{\omega} |\prod y_k \prod z_l|^r.$$

This yields

$$\mathbf{S} \le C \sum_{j=m}^{\lfloor t/\delta \rfloor} \sum_{\omega} |\prod y_k \prod z_l|^r = C \sum_{\omega} \sum_{j=m}^{\lfloor t/\delta \rfloor} |\prod y_k \prod z_l|^r.$$

It follows that (9) will hold if, for all ω,

$$\delta^{1-mr/2} \sum_{j=1}^{\lfloor t/\delta \rfloor} |\prod y_k \prod z_l|^r = o_p(1).$$

But this is equivalent to the set of conditions (5)-(7).

3.2.2 Central limit theorem

In the IA setting, for CLT we are assuming that $r \le 1$. It will be seen, from the examples to be discussed in the next Section, that the restriction to $r \le 1$ is essentially necessary. From (8) we find:

For MPVCLT it suffices that the following conditions are met for $r \le 1$:

$$\delta^{(1-mr)/2}[Z_\delta, ..., Z_\delta]^{[r,...,r]} = o_p(1), \tag{10}$$

$$\delta^{(1-(m-1)r)/2}|\log\delta|^{r/2}[Z_\delta, ..., Z_\delta]^{[r,...,r]}\ [\tbinom{m}{1}] = o_p(1), ... \tag{11}$$

$$\delta^{(1-r)/2}|\log\delta|^{(m-1)r/2}[Z_\delta]^{[r]}\ [\tbinom{m}{m-1}] = o_p(1). \tag{12}$$

For PCLT this reduces to

$$\delta^{(1-r)/2}[Z_\delta]^{[r]} = o_p(1)$$

which can only be satisfied for $r < 1$.

For BPCLT the conditions (in the general $[r,s]$ case) are

$$\delta^{(1-r-s)/2}[Z_\delta, Z_\delta]^{[r,s]} = o_p(1) \tag{13}$$

$$\delta^{(1-r)/2}[Z_\delta, \delta^{-1/2}Y_\delta]^{[r,s]} = o_p(1) \tag{14}$$

$$\delta^{(1-s)/2}[\delta^{-1/2}Y_\delta, Z_\delta]^{[r,s]} = o_p(1). \tag{15}$$

Due to Lemma 1, sufficient for the relations (14) and (15) are

$$\delta^{(1-r)/2}|\log\delta|^{s/2}[Z_\delta]^{[r]} = o_p(1) \tag{16}$$

$$\delta^{(1-s)/2}|\log\delta|^{r/2}[Z_\delta]^{[s]} = o_p(1). \tag{17}$$

Sufficient for (16) is $0 < r < 1$ and $\sup_\delta [Z_\delta]^{[r]} < \infty$. And similarly for (17).

4 Lévy processes with no continuous component

4.1 Preliminaries on Lévy processes and their small-time behaviour

Lévy processes (e.g. Bertoin (1996) and Sato (1999)) with no continuous component are a versatile class of jump processes. Whether MPVCiP or MPVCLT hold, depends on the characteristics of the Lévy process. Notably the number of small jumps is important. We have seen that finite activity restricts $\max r < 2$ and $\max r < 1$, respectively for MPVCiP and MPVCLT. We will get further restrictions, in general, when we have IA.

Let Z_t denote a Lévy process with no continuous component. It incorporates jumps $(\Delta Z_t)_{t\geq 0}$ whose Lévy measure we will write as Π. Π is a Radon measure on $\mathrm{R}^* = \mathrm{R} - \{0\}$ with

$$\int_{\mathrm{R}^*} \left(|x|^2 \wedge 1\right) \Pi(\mathrm{d}x) < \infty. \tag{18}$$

If the stronger condition $\int_{\mathrm{R}^*}(|x| \wedge 1)\Pi(\mathrm{d}x) < \infty$ holds, then we can write

$$Z_t = \sum_{s\leq t} \Delta Z_s \quad \text{and} \quad \mathrm{E}(\exp\{i\lambda Z_t\}) = \exp\{-t\Psi(\lambda)\}, \quad \text{where } \Psi(\lambda) = \int_{\mathrm{R}^*} (1 - e^{i\lambda x})\Pi(\mathrm{d}x),$$

and Z has paths of locally bounded variation. If $\int_{\mathrm{R}^*}(|x| \wedge 1)\Pi(\mathrm{d}x) = \infty$, we allow an additional drift parameter $a \in \mathrm{R}$ so that

$$\mathrm{E}(\exp\{i\lambda Z_t\}) = \exp\{-t\Psi(\lambda)\}, \quad \text{where } \Psi(\lambda) = -i\lambda a + \int_{\mathrm{R}^*} (1 - e^{i\lambda x} + i\lambda x 1_{\{|x|\leq 1\}})\Pi(\mathrm{d}x),$$

and in this case Z has paths of locally unbounded variation.

We define an index

$$\alpha = \inf\left\{\gamma \geq 0 : \int_{[-1,1]} |x|^\gamma \Pi(\mathrm{d}x) < \infty\right\} \in [0, 2].$$

The number α measures how heavily infinite Π is at zero, i.e. how many small jumps Z has.

If Z has bounded variation, then $0 \leq \alpha \leq 1$. If Z has unbounded variation, then $1 \leq \alpha \leq 2$. The boundary $\alpha = 1$ is attained for both bounded and unbounded variation processes. $\Pi(\mathrm{d}x) = |x|^{-2}|\log|x/2||^{-1-\beta}1_{[-1,1]}(x)\mathrm{d}x$ is an example for a bounded variation process with $\alpha = 1$.

The index α can be seen to be greater than or equal (usually equal) to the Blumenthal and Getoor (1961) upper index

$$\alpha^* = \inf\{\gamma \geq 0 : \limsup_{\lambda\to\infty} |\Psi(\lambda)|/\lambda^\gamma = 0\} \in [0, 2].$$

Without loss of generality we can decompose Z into $Z_t = Z_t^{(1)} + Z_t^{(2)}$, where $Z^{(1)}$ and $Z^{(2)}$ are independent processes and $Z^{(2)}$ is defined as

$$Z_t^{(2)} = \sum_{s\leq t} \Delta Z_s I\left(|\Delta Z_s| > 1\right).$$

Clearly $Z^{(2)}$ is a compound Poisson process, and hence of finite activity. The effect of $Z^{(2)}$ on MPVCiP and MPVCLT was studied in the previous Section and so from now on in this Section we can, without loss of generality, set $Z^{(2)}$ to zero, i.e. assume Π is concentrated on $[-1, 1]$.

Lemma 2 *Let Z be a Lévy process with no continuous component and index α. Then*

$$\sup_{\delta>0} \frac{\mathrm{E}\,|Z_\delta|^\gamma}{\delta} < \infty,$$

for all $\alpha < \gamma \leq 1$ if Z has finite mean and bounded variation, and for all $1 \leq \alpha < \gamma \leq 2$ if Z is a zero-mean Lévy process with finite variance.

Proof. Let $\alpha < 1$. From (3) and the compensation formula for Poisson point processes we get for all $\alpha < \gamma \leq 1$

$$\mathrm{E}\,|Z_\delta|^\gamma = \mathrm{E}\left|\sum_{0\leq s\leq\delta} \Delta Z_s\right|^\gamma \leq \mathrm{E}\sum_{0\leq s\leq\delta} |\Delta Z_s|^\gamma = \delta \int_{\mathrm{R}^*} |z|^\gamma \Pi_Z(\mathrm{d}z) < \infty.$$

If $1 \leq \alpha < 2$, we use Monroe embedding $Z_t = B_{T_t}$ into a Brownian motion B, for a subordinator T_t of stopping times for B, with $\mathrm{E}(T_t) = \mathrm{E}\left(Z_t^2\right) < \infty$. Using the explicit embedding of Winkel (2005), we have as Lévy measure of T

$$\Pi_T = \int_{\mathrm{R}^*} \rho_{|x|}\Pi(\mathrm{d}x) + \int_{\mathrm{R}^*}\int_0^{|x|} \frac{|x|}{y^2}\rho_{|x|} * \rho_{|x|}\mathrm{d}y\Pi(\mathrm{d}x),$$

where ρ_x is the distribution of the first passage time at x of a three-dimensional Bessel process starting from zero. In particular, $R_x \sim \rho_x$ has first moment $\mathrm{E}(R_x) = x^2/3$, so that for all $2 \geq \gamma > \alpha \geq 1$, by Jensen's inequality,

$$\int_{\mathrm{R}^*} \mathrm{E}(R_{|x|}^{\gamma/2})\Pi(\mathrm{d}x) \leq \int_{\mathrm{R}^*} \left(\mathrm{E}(R_{|x|})\right)^{\gamma/2}\Pi(\mathrm{d}x) = \left(\frac{1}{3}\right)^{\gamma/2} \int_{\mathrm{R}^*} |x|^\gamma \Pi(\mathrm{d}x) < \infty,$$

and similarly

$$\begin{aligned}\int_{\mathrm{R}^*}\int_0^{|x|} \frac{|x|}{y^2}\mathrm{E}((R_{|y|} + \tilde{R}_{|y|})^{\gamma/2})\mathrm{d}y\Pi(\mathrm{d}x) &\leq \left(\frac{2}{3}\right)^{\gamma/2} \int_{\mathrm{R}^*}\int_0^{|x|} \frac{|x|}{y^2} y^\gamma \mathrm{d}y\Pi(\mathrm{d}x) \\ &= \left(\frac{2}{3}\right)^{\gamma/2} \frac{1}{\gamma-1}\int_{\mathrm{R}^*} |x|^\gamma \Pi(\mathrm{d}x) < \infty.\end{aligned}$$

The sum of the left hand sides is $\int_{(0,\infty)} |z|^{\gamma/2}\Pi_T(\mathrm{d}z)$, so that the index of T is (at most) $\alpha/2$.

Now we invoke Revuz and Yor (1999, Exercise V.(1.23)): $\mathrm{E}\,|B_\tau|^{2p} \leq C_p \mathrm{E}\left(\tau^p\right)$, for all (bounded, but then all) stopping times τ with $\mathrm{E}\left(\tau^p\right) < \infty$, all $p > 0$, and universal constants C_p; see also Revuz and Yor (1999, Theorem IV.(4.10)). This implies $\mathrm{E}\,|Z_\delta|^\gamma = \mathrm{E}\,|B_{T_\delta}|^\gamma \leq C_p \mathrm{E} T_\delta^{\gamma/2}$ and an application of the bounded variation case to the subordinator T completes the proof. ■

4.2 General results on multipower variation for $\mathcal{BSM}$ plus Lévy

We recall that we are working with $X = Y + Z$, where $Y \in \mathcal{BSM}$. No assumptions are made regarding dependence between Y and Z. We can now show the following general result

Theorem 1 *Let Z be a no continuous component Lévy process with index $\alpha \in [0,2]$. Then (i) $0 < r < 2$ $\Rightarrow$ PCiP is valid, (ii) $\alpha < 2$ and $0 < \max r < 2 \Rightarrow$ MPVCiP is valid, (iii) $\alpha < 1$ and $\alpha/(2-\alpha) < r < 1$ $\Rightarrow$ PCLT is valid, (iv) $\alpha < 1$ and $\alpha/(2-\alpha) < \min r \leq \max r < 1 \Rightarrow$ MPVCLT is valid.*

Proof. For the PCiP, note that $\Psi(\lambda)/\lambda^2 \to 0$ as $\lambda \to \infty$ since we have no Gaussian coefficient (cf. Bertoin (1996, Proposition I.2)). Therefore

$$\mathrm{E}\left(\exp\left\{i\lambda\frac{Z_\delta}{\delta^{1/2}}\right\}\right) = \exp\left\{-\delta\Psi\left(\frac{\lambda}{\delta^{1/2}}\right)\right\} \to 1$$

i.e. $Z_\delta/\delta^{1/2} \to 0$ in probability as $\delta \downarrow 0$. Since also $E(Z_\delta^2) = c\delta$, we have that $(Z_\delta/\delta^{1/2})_{\delta>0}$ is bounded in L^2, i.e. convergent in L^r, $1 \leq r < 2$, and it is easily seen that this extends to $0 < r < 2$ (e.g. by raising $Z_\delta/\delta^{1/2}$ to a small power and applying the argument again). Therefore

$$\mathrm{E}\left(\delta^{1-r/2}[Z_\delta]_t^{[r]}\right) = \delta\left\lfloor t/\delta\right\rfloor\frac{\mathrm{E}|Z_\delta|^r}{\delta^{r/2}} \to 0.$$

By (5) PCiP follows. For MPVCiP the argument works for (5) holds by independent increments as

$$\mathrm{E}\left(\delta^{1-r_+/2}[Z_\delta,\ldots,Z_\delta]_t^{[\mathbf{r}]}\right) = \delta\left\lfloor 1-m+t/\delta\right\rfloor\prod_{j=1}^{m}\frac{\mathrm{E}|Z_\delta|^{r_j}}{\delta^{r_j/2}} \to 0,$$

but fails for (6)-(7) because of the log-terms e.g. in (7). However, if $\alpha < 2$, we can adapt the argument as follows. By Lemma 2, we have $\sup_{\delta>0}\left(\delta^{-1}\mathrm{E}|Z_\delta|^\gamma\right) < \infty$ for all $\alpha^* \leq \alpha < \gamma \leq 2$. As above, we have $Z_\delta/\delta^{1/\gamma} \to 0$ in probability, and hence in L^r for $r < \gamma$. This allows us to check (7) for $0 < r_j < \gamma$:

$$\mathrm{E}\left(\delta^{1-r_j/2}\left(\log\left(\frac{1}{\delta}\right)\right)^{r_+-r_j}[Z_\delta]_t^{[r_j]}\right) = \delta\left\lfloor t/\delta\right\rfloor\frac{\mathrm{E}|Z_\delta|^{r_j}}{\delta^{r_j/2}\left(\log\left(\frac{1}{\delta}\right)\right)^{r_j-r_+}} \leq \delta\left\lfloor t/\delta\right\rfloor\frac{\mathrm{E}|Z_\delta|^{r_j}}{\delta^{r_j/\gamma}} \to 0,$$

and similarly all (5)-(7).

For the MPVCLT note that $\alpha < 1$ implies that Z has bounded variation. Furthermore, we can assume that Z has no drift, as this can be placed in the Y process. Now, Lemma 2 gives the basis for the above MPVCiP argument to apply here, for $\alpha < \gamma < 1$, and we can check (12):

$$\mathrm{E}\left(\delta^{1/2-r_j/2}\left(\log\left(\frac{1}{\delta}\right)\right)^{r_+-r_j}[Z_\delta]_t^{[r_j]}\right) = \delta\left\lfloor t/\delta\right\rfloor\frac{\mathrm{E}|Z_\delta|^{r_j}}{\delta^{r_j/2+1/2}\left(\log\left(\frac{1}{\delta}\right)\right)^{r_j-r_+}} \leq \delta\left\lfloor t/\delta\right\rfloor\frac{\mathrm{E}|Z_\delta|^{r_j}}{\delta^{r_j/\gamma}} \to 0$$

if and only if $r_j/2 + 1/2 < r_j/\gamma$, i.e. $r_j > \gamma/(2-\gamma) \downarrow \alpha/(2-\alpha)$ as $\gamma \downarrow \alpha$. It is now easy to repeat the argument and check that then also the remaining equations in (10-12) hold. ■

Apart from a finer distinction on the boundaries such as $\alpha = 2$ or $r = \alpha/(2-\alpha)$ in terms of powers of logs or integral criteria, we believe that the ranges for α and r cannot be extended.

4.3 Examples

In the examples we shall discuss Z is a Lévy jump process and $r_1 = ... = r_m = r$. However, as will be noted at the end of this Section, quite similar results hold for Z being a process of OU type.

Example 1 Suppose Z is the $\Gamma(\nu, \lambda)$ subordinator, i.e. Z is the Lévy process for which the probability density of Z_1 is $\lambda^\nu x^{\nu-1} e^{-\lambda x}/\Gamma(\nu)$. This has IA and $\alpha = 0$ as its index. Consequently: (i) MPVCiP is valid for all $m = 1, 2, ...$ and $0 < r < 2$. (ii) MPVCLT is valid for all $m = 1, 2, ...$ and $0 < r < 1$. However, BPVCLT does not hold if $r = 1$ and $Y \perp\!\!\!\perp Z$.

Example 2 Let Z be the $IG(\phi, \gamma)$ subordinator, i.e. Z is the Lévy process for which the probability density Z_1 is $\delta\,(2\pi)^{-1/2}\, e^{\delta\gamma} x^{-3/2} e^{-\frac{1}{2}(\phi^2 x^{-1} + \gamma^2 x)}$. Again, this has IA, with $\alpha = 1/2$. Consequently: (i) MPVCiP is valid for all $m = 1, 2, ...$ and $0 < r < 2$. (ii) MPVCLT is valid for all m if $\frac{1}{3} = \frac{\alpha}{2-\alpha} < r < 1$. In particular, MPVCLT holds for tripower variation with $r = 2/3$.

Example 3 Let Z be a Variance Gamma Lévy process with parameters ν and λ. This means it can be written as $Z_t = B_{T_t}$, where B is Brownian motion and T is a $\Gamma(\nu, \lambda)$ subordinator, while $B \perp\!\!\!\perp T$. Here $\alpha = 0$ and consequently we have the same conclusion as in Example 1.

Example 4 Let Z be the $NIG(\gamma, 0, 0, \phi)$ Lévy process. This is representable as the subordination of a Brownian motion B by the $IG(\phi, \gamma)$ subordinator. Hence, $\alpha = 2 \times \frac{1}{2} = 1$. Consequently: (i) MPVCiP is valid for all $m = 1, 2, ...$ and $0 < r < 2$. (ii) MPVCLT does not hold for any value of r.

Remark 1 *Suppose Z is an OU process V with a background driving Lévy process (BDLP) L. Letting $V_t^* = \int_0^t V_s \mathrm{d}s$ we have, since V is the solution of $\mathrm{d}V_t = -\lambda V_t + \mathrm{d}L_{\lambda t}$, that $V_t = V_0 - \lambda V_t^* + L_{\lambda t}$. Hence, letting $Y' = Y + V_0 - \lambda V^*$ we see that Y' satisfies the condition (2). Therefore the asymptotics are the same whether $Z = V$ or $Z = L$. In the latter case we are back in the setting of the above examples, where we now apply Theorem 1 to the dependent processes Y' and L.*

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