Volatility scaling's impact on the Sharpe ratio*

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Abstract

We study the econometric properties of dynamic risk parity, which volatility scales to equalise risk through time using the precision process, the inverse of the time-varying volatility, that is σ_t^{-1}. A particular focus is on the impact of the Sharpe ratio. We give necessary and sufficient conditions that volatility scaling improves the Sharpe ratio of an investment. We approximate the Sharpe improvement using the sum of two terms: one determined by the convexity of the precision and the other the covariance of the precision and conditional mean. We show that empirically this approximation is very accurate and we document the relative importance of the two terms.

Keywords: asset allocation; precision process; risk parity; Sharpe ratio; time-varying volatility.

1 Introduction

Taking investment positions that scale inversely proportionally to the risk of the underlying asset is at the heart of "risk parity". Maillard et al. (2010) study the mathematics of risk parity while Asness et al. (2012) discuss some aspects of the empirics. A survey is provided by Roncalli (2014).

Some trading strategies look at a variant of this idea, trying to balance out risk through time rather than across assets. We refer to this as "dynamic risk parity", but it could also be called "volatility scaling". Relevant papers include Hocquard et al. (2013) and Harvey et al. (2018) which study some of the empirical properties of this approach. See also Dopfel and Ramkumar (2013), Barroso and Santa-Clara (2015), Daniel and Moskowitz (2016) and Moreira and Muir (2017). Somewhat further afield, Fleming et al. (2001) and Fleming et al. (2003) look empirically at using volatility predictions in mean-variance asset allocation, comparing to static portfolios.

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The latter paper uses realized volatility type measures (e.g. Andersen et al. (2001) and Barndorff-Nielsen and Shephard (2002)) to improve the forecasts.

In this paper we formalize some of the econometrics of dynamic risk parity, looking at the properties of this scheme. In particular we give necessary and sufficient conditions for when volatility scaling improves the unconditional Sharpe ratio compared to a buy and hold position. Much of this analysis is based on the properties of σ_t^{-1}, which we call the (root) "precision process" or precision process for short. We also approximate the unconditional Sharpe improvement using the sum of two terms: one determined by the convexity of the precision and the other the covariance of the precision and conditional mean. We show that empirically this approximation is very accurate and we document the relative importance of the two terms.

The outline of the rest of this paper is as follows. Section 2 sets up some basic assumptions. Section 3 derives the core properties of volatility scaled returns. These include the first two moments and the corresponding Sharpe ratio, which is compared to the usual buy and hold Sharpe ratio. We give necessary and sufficient conditions for volatility scaling to improve the unconditional Sharpe ratio. The section establishes a very simple and highly accurate approximation to the Sharpe ratio of the volatility scaled position. Section 4 presents an empirical analysis of these results using a large dataset of returns from futures and forward markets. Section 5 concludes.

2 Basic assumptions

Let Y_t be the t-th period return from an investment in a speculative asset (note the investment could itself be the shorting of a futures contract or a cash equity) above the funding cost of the investment (e.g. risk free rate). Write

$$\mathrm{E}(Y_t|\mathcal{F}_{t-1}) = \mu_t, \quad \mathrm{Var}(Y_t|\mathcal{F}_{t-1}) = \sigma_t^2, \quad t = 1, 2, ..., T. \tag{1}$$

We assume $\sigma_t^2 < \infty$ for all t. Here $\mathcal{F}_{t-1}$ is some past data (which could include high frequency data, e.g. Shephard and Sheppard (2010), Hansen et al. (2011)) or a latent process (which could include a volatility process, e.g. the review in Shephard (2005)).

Let

$$\begin{aligned} \mu &= \mathrm{E}(Y_t) = \mathrm{E}(\mu_t), \\ \sigma^2 &= \mathrm{Var}(Y_t) = \mathrm{E}(\sigma_t^2) + \mathrm{Var}(\mu_t), \quad \text{assuming } \mathrm{E}(\sigma_t^2) < \infty, \quad \mathrm{Var}(\mu_t) < \infty. \end{aligned}$$

The buy and hold unconditional Sharpe (1966) ratio is, assuming $\mathrm{E}(\sigma_t^2) < \infty$,

$$S = \frac{\mathrm{E}(Y_t)}{\sqrt{\mathrm{Var}(Y_t)}} = \frac{\mu}{\sigma}.$$

Without any additional loss of generality we can decompose

$$Y_t = \mu_t + \varepsilon_t \sigma_t, \quad \mathrm{E}(\varepsilon_t | \mathcal{F}_{t-1}) = 0, \quad \mathrm{Var}(\varepsilon_t | \mathcal{F}_{t-1}) = 1.$$

Thus $\{\varepsilon_t\}$ is a martingale difference sequence (e.g. Hall and Heyde (1980)) with respect to $\{\mathcal{F}_t\}$ as $\mathrm{E}(|\varepsilon_t|) < \infty$ due to $\sigma^2 < \infty$.

3 Properties of volatility scaling

3.1 Volatility exposure

Volatility scaling delivers the sequence of returns

$$\frac{Y_t}{\sigma_t}.$$

We call σ_t^{-1} the "root precision process" or precision process for short. It plays the central role in the results in this note. The return sequence has conditional variances

$$\mathrm{Var}\left(\frac{Y_t}{\sigma_t} | \mathcal{F}_{t-1} \right) = 1, \quad t = 1, 2, ..., T,$$

which is constant through time and this is a dynamic version of "risk parity".

This approach has some appeal for it allows an investor to scale up or down the position to deliver, say, \$100 standard deviation of risk each day.

Theorem 1 states the main properties of the resulting volatility scaled position.

Theorem 1 *If (1) holds and* $\left|\mathrm{E}(\sigma_t^{-1}\mu_t)\right| < \infty$, *then*

$$\mathrm{E}\left(\frac{Y_t}{\sigma_t} \right) = \mathrm{E}\left(\sigma_t^{-1}\mu_t \right) = \mu \mathrm{E}\left(\sigma_t^{-1} \right) + \mathrm{Cov}(\sigma_t^{-1}, \mu_t).$$

If $\left|\mathrm{E}(\sigma_t^{-2}\mu_t^2)\right| < \infty$, *then*

$$\mathrm{Var}\left(\frac{Y_t}{\sigma_t} \right) = 1 + \mathrm{Var}\left(\sigma_t^{-1}\mu_t \right).$$

If $\sigma < \infty$, *then define*

$$\xi_0 = \mathrm{E}\left(\sigma\sigma_t^{-1} \right), \quad \xi_1 = \mathrm{Cov}(\sigma\sigma_t^{-1}, \mu^{-1}\mu_t).$$

Then

$$\begin{aligned}
\mathrm{E}\left(\frac{Y_t}{\sigma_t} \right) &= \frac{\mu}{\sigma}\gamma_1, \quad \text{where} \quad \gamma_1 = \mathrm{E}\left(\frac{\sigma}{\sigma_t}\frac{\mu_t}{\mu} \right) = \xi_0 + \xi_1, \\
\mathrm{Var}\left(\frac{Y_t}{\sigma_t} \right) &= 1 + \frac{\mu^2}{\sigma^2}\gamma_2, \quad \gamma_2 = \mathrm{Var}\left(\frac{\sigma}{\mu}\frac{\mu_t}{\sigma_t} \right) = \mathrm{E}\left(\frac{\sigma^2}{\sigma_t^2}\frac{\mu_t^2}{\mu^2} \right) - \gamma_1^2.
\end{aligned}$$

Proof.

$$\begin{aligned}
\mathrm{E}\left(\frac{Y_t}{\sigma_t}\right) &= \mathrm{E}\left\{\mathrm{E}\left(\frac{Y_t}{\sigma_t}|\mathcal{F}_{t-1}\right)\right\}, \quad \text{by iterative expectation} \\
&= \mathrm{E}\left(\sigma_t^{-1}\mu_t\right), \quad \text{as } \left|\mathrm{E}(\sigma_t^{-1}\mu_t)\right| < \infty \\
&= \mathrm{E}\left(\sigma_t^{-1}\right)\mathrm{E}\left(\mu_t\right) + \mathrm{Cov}\left(\sigma_t^{-1}, \mu_t\right) = \mu \mathrm{E}\left(\sigma_t^{-1}\right) + \mathrm{Cov}(\sigma_t^{-1}, \mu_t).
\end{aligned}$$

Likewise

$$\begin{aligned}
\mathrm{Var}\left(\frac{Y_t}{\sigma_t}\right) &= \mathrm{E}\left\{\mathrm{Var}\left(\frac{Y_t}{\sigma_t}|\mathcal{F}_{t-1}\right)\right\} + \mathrm{Var}\left\{\mathrm{E}\left(\frac{Y_t}{\sigma_t}|\mathcal{F}_{t-1}\right)\right\} \\
&= \mathrm{E}(1) + \mathrm{Var}\left(\sigma_t^{-1}\mu_t\right) = 1 + \mathrm{Var}\left(\sigma_t^{-1}\mu_t\right).
\end{aligned}$$

Rewriting the results completes our proof.

We call γ_1 the "volatility mean exposure."

Convexity of the precision process σ_t^{-1} delivers an important bound on a scale invariant measure.

Lemma 1 *If* $\mathrm{E}(\sigma_t^2) < \infty$ *and* $\left|\mathrm{E}(\sigma_t^{-2}\mu_t^2)\right| < \infty$, *then*

$$\xi_0 = \mathrm{E}\left(\sigma\sigma_t^{-1}\right) \geq \left\{1 - \sigma^{-2}\mathrm{Var}(\mu_t)\right\}^{-1/2} = \left\{1 - \frac{\mu^2}{\sigma^2}\mathrm{Var}(\mu^{-1}\mu_t)\right\}^{-1/2} \geq 1.$$

Proof. Now $\mathrm{E}\left(\sigma_t^{-1}\right) < \infty$ so

$$\mathrm{E}\left(\sigma_t^{-1}\right) = \mathrm{E}\left(\lambda_t^{-1/2}\right), \quad \lambda_t = \sigma_t^2, \quad \sigma^2 = \mathrm{E}\left(\lambda_t\right) + \mathrm{Var}(\mu_t),$$

but $\partial^2\lambda_t^{-1/2}/\partial\lambda_t^2 > 0$ so $\lambda_t^{-1/2}$ is convex in λ_t. Then by Jensen's inequality

$$\mathrm{E}\left(\sigma_t^{-1}\right) = \mathrm{E}\left(\lambda_t^{-1/2}\right) \geq \left(\mathrm{E}\lambda_t\right)^{-1/2} = \sigma^{-1}\left\{1 - \sigma^{-2}\mathrm{Var}(\mu_t)\right\}^{-1/2},$$

as $\mathrm{E}\left(\lambda_t\right) = \sigma^2 - \mathrm{Var}(\mu_t)$, so

$$\xi_0 \geq \left\{1 - \sigma^{-2}\mathrm{Var}(\mu_t)\right\}^{-1/2}.$$

This completes our proof.

Remark 1 *Suppose we lever a volatility scaled position to deliver investment returns*

$$\sigma\sigma_t^{-1}Y_t,$$

so

$$\mathrm{Var}(\sigma\sigma_t^{-1}Y_t|\mathcal{F}_{t-1}) = \sigma^2 = \mathrm{Var}(Y_t),$$

the unconditional variance of a buy and hold investment Y_t. *Thus we can think of* $\sigma\sigma_t^{-1}$ *as inducing a particularly interesting level of time-varying financial leverage, i.e. if* $\sigma\sigma_t^{-1}$ *is 2 at time* t *then the scaled position at time* t *is twice as large as the corresponding buy and hold. Recall*

$$\gamma_1 = \xi_0 + \xi_1, \quad \textit{where} \quad \xi_0 = \mathrm{E}\left(\sigma\sigma_t^{-1}\right), \quad \xi_1 = \mathrm{Cov}(\sigma\sigma_t^{-1}, \mu^{-1}\mu_t).$$

Here ξ_0 measures the mean of $\sigma\sigma_t^{-1}$. This is a convexity term. Likewise ξ_1 measures the timing of covariation of $\sigma\sigma_t^{-1}$ with $\mu^{-1}\mu_t$. Both ξ_0 and ξ_1 are invariant to the investment being levered up by a time-invariant amount, and so in particular do not change as we move from the position being long to being short. Finally, we note that $\mathrm{E}(Y_t\sigma_t^{-1}) = \mathrm{E}(\mu_t\sigma_t^{-1})$ *as ε_t is a martingale difference sequence. Thus* $\mathrm{Cov}\left(\sigma_t^{-1}, Y_t\right) = \mathrm{Cov}(\sigma_t^{-1}, \mu_t)$. *So ξ_1 can be estimated simply. Likewise* $\mathrm{E}(Y_t^2\sigma_t^{-2}) = \mathrm{E}(\mu_t^2\sigma_t^{-2}) + 1$ *and so* $\mathrm{Var}\left(\sigma_t^{-1}\mu_t\right) = \mathrm{Var}(Y_t\sigma_t^{-1}) - 1$. *These equalities are just rewrites of parts of Theorem 1.*

A simple summary of investment behaviour of positions is through the unconditional Sharpe ratio. Here we derive its basic properties. In addition we give necessary and sufficient conditions for the Sharpe ratio of the volatility scaled position being larger than the Sharpe of a buy and hold position.

Theorem 2 *The unconditional Sharpe ratio of the volatility scaled position exists if* $\mathrm{Var}\left(\sigma_t^{-1}\mu_t\right) < \infty$. *It is*

$$S_\sigma = \frac{\mathrm{E}\left(\sigma_t^{-1}\mu_t\right)}{\sqrt{1 + \mathrm{Var}\left(\sigma_t^{-1}\mu_t\right)}}.$$

If, in addition, $\sigma^2 < \infty$, then this unconditional Sharpe ratio can be rewritten as

$$S_\sigma = S\frac{\gamma_1}{\left(1 + S^2\gamma_2\right)^{1/2}}, \quad \textit{recalling} \quad S = \frac{\mu}{\sigma}.$$

So $S_\sigma \geq S$ if and only if

$$\gamma_1 \geq \sqrt{1 + S^2\gamma_2} = \sqrt{1 + \mathrm{Var}\left(\sigma_t^{-1}\mu_t\right)}.$$

Proof. The first result uses Theorem 1, so the Sharpe ratio is

$$S_\sigma = \frac{\frac{\mu}{\sigma}\gamma_1}{\left(1 + \frac{\mu^2}{\sigma^2}\gamma_2\right)^{1/2}},$$

so we need for $S_\sigma \geq S$ that

$$\gamma_1 = \mathrm{E}\left(\frac{\sigma}{\sigma_t}\frac{\mu_t}{\mu}\right) \geq \left(1 + \frac{\mu^2}{\sigma^2}\gamma_2\right)^{1/2} = \sqrt{1 + \mathrm{Var}\left(\sigma_t^{-1}\mu_t\right)}.$$

Then recalling $\mathrm{E}(\mu_t) = \mu$ we have the final result using the definition of a covariance.

This completes our proof.

Remark 2 *A potentially useful approximation of the unconditional Sharpe ratio is obtained using a Taylor expansion of the denominator,*

$$S_\sigma = \frac{\frac{\mu}{\sigma}\gamma_1}{\left(1 + \frac{\mu^2}{\sigma^2}\gamma_2\right)^{1/2}} \simeq S\gamma_1\left(1 - \frac{S^2}{2}\gamma_2\right), \quad \textit{recalling} \quad S = \frac{\mu}{\sigma}.$$

It will turn out that the approximation $S_\sigma \simeq S\gamma_1$ is empirically compelling.

3.2 Comparison with buy and hold

Volatility scaling delivers, compared to a static buy and hold, returns

$$R_t = \left(\frac{Y_t}{\sigma_t} - \frac{Y_t}{\sigma}\right) = \frac{Y_t}{\sigma}\left(\frac{\sigma}{\sigma_t} - 1\right), \quad \text{so} \quad \mathrm{E}(R_t) = \frac{\mu}{\sigma}(\xi_0 - 1) + \mathrm{Cov}(\sigma_t^{-1}, \mu_t).$$

Hence volatility scaling delivers expected "excess returns" compared to a static buy and hold iff $\gamma_1 > 1$ or

$$\xi_0 > 1 - \mathrm{Cov}(\sigma\sigma_t^{-1}, \mu^{-1}\mu_t) = 1 - \xi_1. \tag{2}$$

Finally, Dachraoui (2018) has a convexity argument that volatility scaling delivers excess returns. But Dachraoui uses Jensen's inequality to show that $\mathrm{E}\left(\sigma_t^{-1}\right) \geq \{\mathrm{E}(\sigma_t)\}^{-1}$. But this inequality is not our focus recalling $\mathrm{E}(\sigma_t) \neq \sigma$.

3.3 When $\mu_t = \mu$

If we think of $\mu_t = \mu$ then the results simplify to

$$\begin{aligned} \mathrm{E}\left(\frac{Y_t}{\sigma_t}\right) &= \mu \mathrm{E}\left(\sigma_t^{-1}\right), \quad \mathrm{Var}\left(\frac{Y_t}{\sigma_t}\right) = 1 + \mu^2 \mathrm{Var}\left(\sigma_t^{-1}\right), \quad \xi_1 = 0, \\ \mathrm{E}\left(\frac{Y_t}{\sigma_t} - \frac{Y_t}{\sigma}\right) &= \frac{\mu}{\sigma}(\xi_0 - 1), \quad S_\sigma = S\xi_0\left\{1 + S^2 \mathrm{Var}\left(\sigma\sigma_t^{-1}\right)\right\}^{-1/2}. \end{aligned}$$

Hence volatility scaling delivers excess returns compared to buy and hold as $\xi \geq 1$, while the volatility scaling Sharpe ratio is $S_\sigma \geq S$, which is the Sharpe ratio of the buy and hold, if and only if

$$\xi_0 \geq \sqrt{1 + \mu^2 \mathrm{Var}\left(\sigma_t^{-1}\right)}.$$

The approximation to the volatility scaling unconditional Sharpe ratio is

$$S_\sigma \simeq S\xi_0\left\{1 - \frac{S^2}{2}\mathrm{Var}\left(\sigma\sigma_t^{-1}\right)\right\}, \quad \text{recalling} \quad S = \frac{\mu}{\sigma}.$$

When the daily Sharpe ratio is tiny, this looks roughly like $\frac{\mu}{\sigma}\xi_0 \geq \frac{\mu}{\sigma}$. Thus the volatility scaled position should have a higher Sharpe ratio. However, if $\mathrm{Var}\left(\sigma\sigma_t^{-1}\right)$ is enormous then the Sharpe ratio can be lower.

Remark 3 *The results that the Sharpe ratio is typically higher for the volatility scaled positions is not surprising as the conditional mean is constant and so the conditional Sharpe ratio is higher when the conditional volatility is lower. Hence volatility scaling magnifies exposure when the Sharpe ratio is high and reduces it when it is low. However, our result shows this is not always true.*

3.4 When $\mu_t = c\sigma_t$

Suppose $\mu_t = c\sigma_t$, so the conditional Sharpe μ_t/σ_t is constant. Then, whatever the value of c,

$$\begin{aligned} S &= \frac{c\mathrm{E}(\sigma_t)}{\sigma}, \quad \gamma_1 = \mathrm{E}\left(\frac{\sigma}{\sigma_t}\frac{\mu_t}{\mu}\right) = \frac{\sigma}{\mathrm{E}(\sigma_t)}, \quad \gamma_2 = \mathrm{Var}(\sigma\sigma_t^{-1}\mu^{-1}\mu_t) = 0, \quad \text{so} \\ S_\sigma &= S\gamma_1 = c. \end{aligned}$$

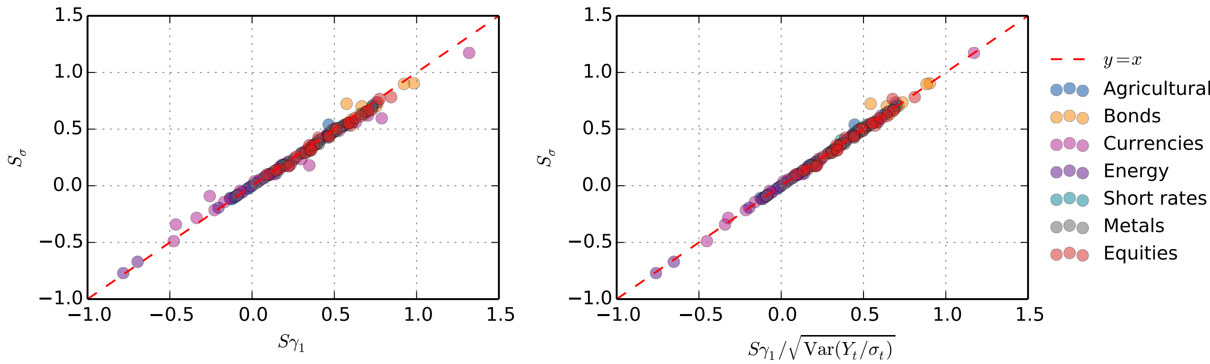

Figure 1: Left hand side shows the cross plot of the Sharpe ratio for volatility scaled assets, S_σ, against $S\gamma_1$. The $y = x$ line is shown using a dotted line. The cross plot is for 142 different futures and forwards markets. A different plotting symbol is used for each market segment. The right hand side uses a finer approximation, normalizing results by the standard deviation of the scaled position.

Finally $\gamma_1 \geq 1$, so volatility scaling non-decreases the absolute value of the unconditional Sharpe ratio.

To show these results note that $\mathrm{E}(\mu_t) = c\mathrm{E}(\sigma_t)$, while $\mathrm{E}(\sigma_t^{-1}\mu_t) = c$, $\mathrm{Var}(\sigma_t^{-1}\mu_t) = 0$. Then

$$\mathrm{E}\left(\frac{Y_t}{\sigma_t}\right) = \mathrm{E}\left(\sigma_t^{-1}\mu_t\right) = c, \quad \mathrm{Var}\left(\frac{Y_t}{\sigma_t}\right) = 1 + \mathrm{Var}\left(\sigma_t^{-1}\mu_t\right) = 1, \quad \text{so} \quad S_\sigma = c.$$

Further,

$$\begin{aligned} \gamma_1 &= \mathrm{E}\left(\frac{\sigma}{\sigma_t}\frac{\mu_t}{\mu}\right) = \frac{\sigma}{\mathrm{E}(\sigma_t)}, \quad \xi_0 = \mathrm{E}(\sigma\sigma_t^{-1}), \quad \xi_1 = \frac{\sigma}{\mathrm{E}(\sigma_t)} - \mathrm{E}(\sigma\sigma_t^{-1}), \\ \gamma_2 &= \mathrm{Var}(\sigma\sigma_t^{-1}\mu^{-1}\mu_t) = 0. \end{aligned}$$

Finally,

$$\frac{\sigma}{\mathrm{E}(\sigma_t)} = \frac{\sqrt{\mathrm{E}(\sigma_t^2) + c^2\mathrm{Var}(\sigma_t)}}{\mathrm{E}(\sigma_t)} \geq \frac{\sqrt{\mathrm{E}(\sigma_t^2) + c^2\mathrm{Var}(\sigma_t)}}{\sqrt{\mathrm{E}(\sigma_t^2)}} \geq 1.$$

4 Empirical assessment

Figure 1 illustrates the approximation $S_\sigma \simeq S\gamma_1$ by plotting

$$S_\sigma \quad \text{against} \quad S\gamma_1$$

across 142 futures and foreign exchange forwards markets. This is based on 30 years of daily data (1988-2017). Only markets where at least 10 years of data are available are included in our database. Throughout the volatility σ_t was approximated by the square root of an exponentially weighted moving average (EWMA) of past squared returns (i.e. an integrated GARCH model) built using a 12-day half-life:

$$\sigma_t^2 = 0.94\sigma_{t-1}^2 + 0.06Y_{t-1}^2, \quad t = 1, 2, ..., T.$$

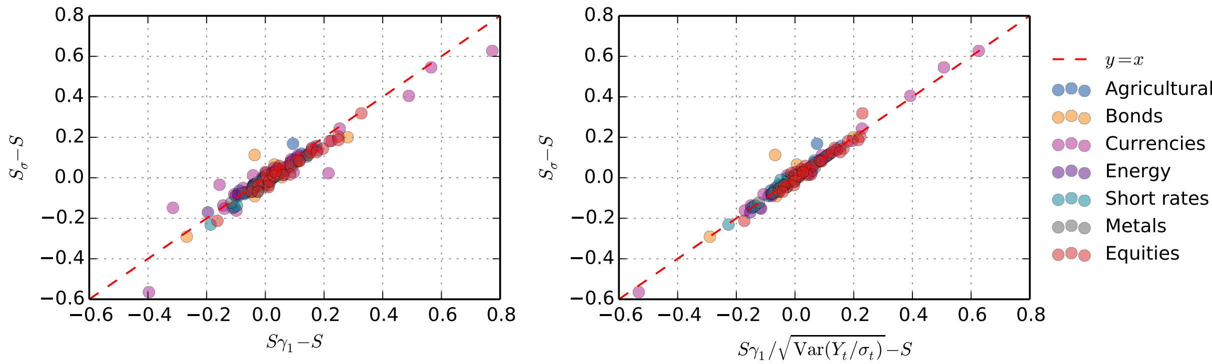

Figure 2: Left hand side shows the cross plot of the Sharpe ratio improvement made by volatility scaling, $S_\sigma - S$, against $S\gamma_1 - S$. The $y = x$ line is shown using a dotted line. The cross plot is for 142 different futures and forwards markets. The right hand side uses a finer approximation.

This suggests that indeed γ_1 is the crucial term in determining the adjustment in the Sharpe ratio created by volatility scaling.

Figure 2 shows how the unconditional Sharpe ratio improves. The left hand side plots $S_\sigma - S$ against $S\gamma_1 - S$, showing that overwhelmingly the improvement is well modelled by using γ_1. The right hand plot shows the corresponding high resolution version.

To understand the drivers of γ_1 for each market we look at its components ξ_0 and ξ_1.

Table 1 provides descriptive statistics of the results, where the measures of variation are across different markets. The top half of the Table shows statistics across the different 142 markets. The bottom half looks at the average estimates of γ_1, ξ_0 and ξ_1 within 7 categories of markets.

	Mean	Median	sd	$Q_{0.1}$	$Q_{0.9}$		
γ_1	1.33	1.08	2.20	0.63	1.77		
ξ_0	1.36	1.26	0.30	1.16	1.64		
ξ_1	-0.03	-0.20	2.21	-0.88	0.52		
				$\mathrm{Cov}(\xi_0, \xi_1)$	=-0.07	$\mathrm{Cor}(\xi_0, \xi_1)$	=-0.11
	Agricultural	Bonds	Currencies	Energy	Equities	Metals	Short rates
Mean γ_1	1.99	1.07	1.46	1.04	1.18	0.93	1.04
Mean ξ_0	1.21	1.29	1.44	1.22	1.35	1.24	1.68
Mean ξ_1	0.78	-0.21	0.02	-0.17	-0.16	-0.31	-0.63
Median γ_1	1.00	1.06	1.33	1.00	1.16	0.99	0.99
Median ξ_0	1.19	1.18	1.30	1.21	1.35	1.23	1.55
Median ξ_1	-0.20	-0.11	-0.01	-0.21	-0.17	-0.24	-0.55
d	22	14	40	8	29	14	15

Table 1: Components of the Sharpe ratio for the scaled position measured over 142 futures and foreign exchange forwards markets. Also given are the mean and median within categories of markets. Q_τ is the τ level estimated quantile. d is the number of markets within each category.

The results show that ξ_0 is typically around 30% higher than 1, hence the convexity of volatility is materially important. It does not particularly change its features across the different categories of markets.

In comparison ξ_1 is much more variable across assets. It is important to put this in some econometric context: it is very hard to estimate ξ_1 precisely as it is driven by the time varying mean. There seems to be a tendency for ξ_1 to be modestly negative, but its imprecision dominates that tendency. The results are clearer for the median, where ξ_1 is modestly negative for all categories.

Overall γ_1 has a mean of around 1.3 and a median of 1.08, so has some assets with very large γ_1 while rarely does γ_1 get small (note that $Q_{0.9} - Q_{0.5} = 0.69$ while $Q_{0.5} - Q_{0.1} = 0.45$).

5 Conclusions

Volatility scaling is a commonly used investment method. Empirical work suggests that sometimes volatility scaling can potentially improve the Sharpe ratio of the returns. In this paper we have filled out the econometrics of these points, providing necessary and sufficient conditions for these results to hold.

In particular we decompose the potential unconditional Sharpe ratio improvements into two components: one due to the convexity of the precision process and the other the covariation of the movements between the precision and time-varying conditional mean. Our empirical analysis suggests that the former is quite stable across market categories, while the latter varies considerably.

This paper has not addressed how to deploy methods to mitigate the effects of trading costs on the performance of volatility scaling. We will report on that topic seperately.

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